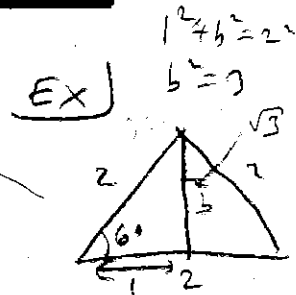
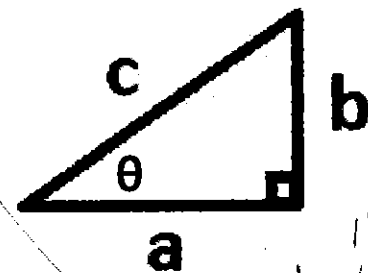


Close Wed: HW_5A,5B,5C (6.5,7.1,7.2)
 Next Wed: HW_6A,6B,6C (7.3,7.4,7.5)
 Work ahead!!! Be done with 7.2 today
 and start to look at 7.3.

What are these in terms of a, b, and c?

$$\begin{aligned} \sin(\theta) &= \frac{b}{c} = \frac{\sqrt{3}}{2} \\ \cos(\theta) &= \frac{a}{c} = \frac{1}{2} \\ \tan(\theta) &= \frac{b}{a} = \frac{\sqrt{3}}{1} \\ \sec(\theta) &= \frac{c}{a} = 2 \end{aligned}$$



7.2 Trig Integrals (continued)

Entry Task: Fill in the blanks

Square Identities	
$\sin^2(x)$	$= 1 - \cos^2(x)$
$\cos^2(x)$	$= 1 - \sin^2(x)$
$\tan^2(x)$	$= \sec^2(x) - 1$
$\sec^2(x)$	$= 1 + \tan^2(x)$
Half Angle Identities	
$\sin^2(x)$	$= \frac{1}{2} (1 - \cos(2x))$
$\cos^2(x)$	$= \frac{1}{2} (1 + \cos(2x))$
$\sin(x) \cos(x)$	$= \frac{1}{2} \sin(2x)$

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = 2 \cos^2(x) - 1$$

Integrate $\int \sin^3(x) \cos^2(x) dx$

$$\begin{aligned} & \int \sin^2(x) \cos^2(x) \sin(x) dx \\ & \int (1 - \cos^2(x)) \cos^2(x) \sin(x) dx \\ & \int (1 - u^2) u^2 (-1) du \\ &= \int -u^2 + u^4 du \\ &= -\frac{1}{3} u^3 + \frac{1}{5} u^5 + C \\ &= \left[-\frac{1}{3} \cos^3(x) + \frac{1}{5} \cos^5(x) + C \right] \end{aligned}$$

$$\begin{aligned} u &= \cos(x) \\ du &= -\sin(x) dx \\ (-1) du &= \sin(x) dx \end{aligned}$$

Integrals involving $\sin(x)$ and $\cos(x)$

Case 1 ($\cos(x)$ or $\sin(x)$ has odd power)

- Separate one from the odd power.

(i.e. pull out one $\sin(x)$ or $\cos(x)$)

- Use $\sin^2(x) = 1 - \cos^2(x)$

$$\cos^2(x) = 1 - \sin^2(x)$$

(get rest in term of the other)

- Use u-substitution.

← BOTH ODD
↓

$$\int \sin^3(x) \cos^5(x) dx$$

$$\int \sin^2(x) \cos^5(x) \sin(x) dx$$

$$\int (1 - \cos^2(x)) \cos^5(x) \sin(x) dx$$

$$u = \cos(x) \\ du = -\sin(x) dx$$

$$\int (1 - u^2) u^5 (-1) du$$

$$\int -u^5 + u^7 du = -\frac{1}{6} u^6 + \frac{1}{8} u^8 + C$$

$$= -\frac{1}{6} \cos^6(x) + \frac{1}{8} \cos^8(x) + C$$

Integrals involving $\sin(x)$ and $\cos(x)$

Case 1 ($\cos(x)$ or $\sin(x)$ has odd power)

- Separate one from the odd power.
(i.e. pull out one $\sin(x)$ or $\cos(x)$)
- Use $\sin^2(x) = 1 - \cos^2(x)$
 $\cos^2(x) = 1 - \sin^2(x)$
(get rest in term of the other)
- Use u-substitution.

Case 2 (both $\sin(x)$, $\cos(x)$ even powers)

- Use

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

over and over again until you can integrate.

Example:

$$\begin{aligned} & \int \cos^4(x) dx \\ &= \int \cos^2(x) \cos^2(x) dx \\ &= \int \frac{1}{2}(1 + \cos(2x)) \frac{1}{2}(1 + \cos(2x)) dx \\ &= \frac{1}{4} \int 1 + 2\cos(2x) + \cos^2(2x) dx \\ &= \frac{1}{4} \int 1 + 2\cos(2x) + \frac{1}{2}(1 + \cos(4x)) dx \\ &= \frac{1}{4} \int \frac{3}{2} + 2\cos(2x) + \frac{1}{2}\cos(4x) dx \\ &= \frac{1}{4} \left[\frac{3}{2}x + \sin(2x) + \frac{1}{8}\sin(4x) \right] + C \\ &= \boxed{\frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C} \end{aligned}$$

Integrals involving $\tan(x)$ and $\sec(x)$

Case 3 ($\sec(x)$ has an even power)

- Separate out $\sec^2(x)$
- Use $\sec^2(x) = 1 + \tan^2(x)$
(get rest in terms of $\tan(x)$)
- Use $u = \tan(x)$

Example: $\int \tan^5(x) \sec^4(x) dx$

$$\int \tan^5(x) \sec^2(x) \sec^2(x) dx$$

$$\int \tan^5(x) (1 + \tan^2(x)) \sec^2(x) dx$$

$$\int u^5 (1 + u^2) du$$

$$u = \tan(x) dx$$

$$du = \sec^2(x) dx$$

$$\int u^5 + u^7 du$$

$$= \frac{1}{6} u^6 + \frac{1}{8} u^8 + C$$

$$= \left[\frac{1}{6} \tan^6(x) + \frac{1}{8} \tan^8(x) + C \right]$$

Case 4 ($\tan(x)$ has an odd power)

AND AT
LEAST ONE $\sec(x)$

- Separate out $\sec(x) \tan(x)$
- Use $\tan^2(x) = \sec^2(x) - 1$
(get rest in terms of $\sec(x)$)
- Use $u = \sec(x)$

Example: $\int \tan^3(x) \sec(x) dx$

$$\int \tan^2(x) \sec(x) \tan(x) dx$$

$$\int (\sec^2(x) - 1) \sec(x) \tan(x) dx$$

$$u = \sec(x)$$

$$du = \sec(x) \tan(x) dx$$

$$\int u^2 - 1 du$$

$$\frac{1}{3} u^3 - u + C$$

$$\boxed{\frac{1}{3} \sec^3(x) - \sec(x) + C}$$

And if you've tried the four cases and are stuck, here are things to try:

1. Rewrite in terms of $\sin(x)$ and $\cos(x)$.
2. Rewrite in terms of $\sec(x)$ and $\tan(x)$.
3. Try using trig identities.

There are still a few "holes".

Namely, when there is one $\tan(x)$ or an odd power on $\sec(x)$. You can quote these (proof in the book):

Ex) $\int \frac{\cos(x) + \sin(2x)}{\sin(x)} dx$ ← to start!

$$= \int \frac{\cos(x) + 2\sin(x)\cos(x)}{\sin(x)} dx$$

$$\int \frac{\tan(x)\cos(x)}{\sec(x)} dx = \int \frac{\frac{\sin(x)}{\cos(x)} \cos(x)}{\frac{1}{\cos(x)}} dx$$

$$= \int \sin(x)\cos(x) dx$$

$$\int \tan(x) dx = \ln|\sec(x)| + C$$

$$\int \sec(x) dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \sec^3(x) dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)| + C$$

7.3 Trigonometric Substitution

Goal: Develop a method to evaluate integrals involving expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$

CASE	SUBSTITUTION
$a^2 - x^2$	$x = a \sin(\theta)$, $-\pi/2 \leq \theta \leq \pi/2$
$a^2 + x^2$	$x = a \tan(\theta)$, $-\pi/2 < \theta < \pi/2$
$x^2 - a^2$	$x = a \sec(\theta)$, $0 \leq \theta < \pi/2$, $\pi \leq \theta < 3\pi/2$

Example:

$$1. \int \frac{x^3}{\sqrt{4-x^2}} dx$$

$x = 2 \sin \theta$
 $dx = 2 \cos \theta d\theta$

$$= \int \frac{8 \sin^3 \theta}{\sqrt{4-4\sin^2 \theta}} 2 \cos \theta d\theta$$

$$= \int \frac{8 \sin^3 \theta}{2 \cos \theta} 2 \cos \theta d\theta$$

$$= 8 \int \sin^2 \theta d\theta$$

KEY
NOTE

$$4 - 4 \sin^2 \theta = 4(1 - \sin^2 \theta)$$

$$= 4 \cos^2 \theta$$

$$\Rightarrow \sqrt{4-4\sin^2 \theta} = \sqrt{4\cos^2 \theta} = 2 \cos \theta$$

$$8 \int \sin^2 \theta \cos \theta d\theta$$

$$8 \int (1 - \cos^2 \theta) \cos \theta d\theta$$

$$8 \int (1 - u^2) (-1) du$$

$$u = \cos \theta$$

$$du = -\sin \theta d\theta$$

$$(-1) du = \sin \theta d\theta$$

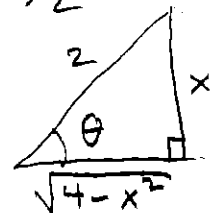
$$-8u + \frac{8}{3}u^3 + C$$

$$-8 \cos \theta + \frac{8}{3} \cos^3 \theta + C$$

$$-8 \frac{\sqrt{4-x^2}}{2} + \frac{8}{3} \left(\frac{\sqrt{4-x^2}}{2} \right)^3 + C$$

$$-4\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} + C$$

$$\sin \theta = \frac{x}{2}$$



$$\cos \theta = \frac{\sqrt{4-x^2}}{2}$$

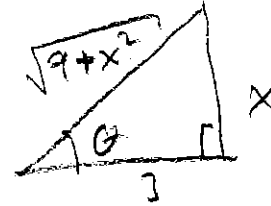
Trigonometric Substitution Method:

- Substitute, don't forget $dx = ??d\theta$.
Simplify (eliminate root)
- Use 7.2 methods for trig integrals.
- Draw a triangle and return to x .

$$2. \int \sqrt{9+x^2} dx$$

$$\boxed{x = 3 \tan \theta}$$
$$dx = 3 \sec^2 \theta d\theta$$

$$\tan \theta = \frac{x}{3}$$



$$\sec \theta = \frac{\sqrt{9+x^2}}{3}$$

$$\int \underbrace{\sqrt{9+9\tan^2\theta}}_{3 \sec \theta} \cdot 3 \sec^2 \theta d\theta$$

$$9 \int \sec^3 \theta d\theta$$

$$\frac{9}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C$$

$$\frac{9}{2} \left(\frac{\sqrt{9+x^2}}{3} \cdot \frac{x}{3} + \ln \left| \frac{\sqrt{9+x^2}}{3} + \frac{x}{3} \right| \right) + C$$

$$= \frac{1}{2} x \sqrt{9+x^2} + \frac{9}{2} (\ln |\sqrt{9+x^2} + x| - \ln(3)) + C$$

$$= \frac{1}{2} x \sqrt{9+x^2} + \frac{9}{2} \ln |\sqrt{9+x^2} + x| + D$$

$$3. \int \frac{\sqrt{x^2 - 16}}{x} dx$$

$$\boxed{x = 4 \sec \theta}$$

$$dx = 4 \sec \theta \tan \theta d\theta$$

$$\sqrt{16(\sec^2 \theta - 1)}$$

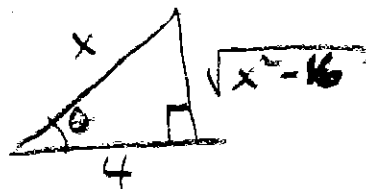
$$= \sqrt{16 \tan^2 \theta}$$

$$= 4 \tan \theta$$

$$\int \frac{\sqrt{16 \sec^2 \theta - 16}}{4 \sec \theta} 4 \sec \theta \tan \theta d\theta$$

$$\int 4 \tan \theta \tan \theta d\theta$$

$$\sec \theta = x/4$$



$$= 4 \int \tan^2 \theta d\theta$$

$$\theta = \sec^{-1}(x/4)$$

$$= 4 \int \sec^2 \theta - 1 d\theta$$

$$= 4 (\tan \theta - \theta) + C$$

$$= 4 \tan \theta - 4\theta + C$$

$$= 4 \frac{\sqrt{x^2 - 16}}{4} - 4 \sec^{-1}(x/4) + C = \boxed{\sqrt{x^2 - 16} - 4 \sec^{-1}(x/4) + C}$$

Completing the Square:

$$\sqrt{ax^2 + bx + c}$$

If you encounter a “**middle term**” (like **bx** above), then complete the square.

Example: $\sqrt{64 - 24x - 4x^2}$

i) **Factor out the “a”.**

$$\sqrt{4(16 - 6x - x^2)} = 2\sqrt{16 - 6x - x^2}$$

ii) **Add/subtract “half-middle squared”**

$$\text{Half of middle} = (-6)/2 = -3$$

$$\text{Squared} = (-3)^2 = 9$$

$$2\sqrt{16 + 9 - 9 - 6x - x^2}$$

iii) **Factor the perfect square**

$$2\sqrt{25 - (x + 3)^2}$$

iv) **Check your work!!!!**

Example:

$$\int \frac{x}{\sqrt{64 - 24x - 4x^2}} dx$$

$$= \int \frac{x}{2\sqrt{25 - (x+3)^2}} dx$$

$$\boxed{x+3 = 5 \sin \theta} \rightarrow x = 5 \sin \theta - 3$$
$$dx = 5 \cos \theta d\theta$$

$$\sin \theta = \frac{x+3}{5}$$

$$= \frac{1}{2} \int \frac{(5 \sin \theta - 3)}{\sqrt{25 - 25 \sin^2 \theta}} 5 \cos \theta d\theta$$

$$= \frac{1}{2} \int \frac{(5 \sin \theta - 3)}{\cancel{5 \cos \theta}} \cancel{5 \cos \theta} d\theta$$

$$= \frac{1}{2} (-5 \cos \theta - 3\theta) + C$$

$$= -\frac{5}{2} \cos \theta - \frac{3}{2} \theta + C$$

$$= \boxed{-\frac{5}{2} \frac{\sqrt{25 - (x+3)^2}}{5} - \frac{3}{2} \sin^{-1}\left(\frac{x+3}{5}\right) + C}$$

